KUWAIT UNIVERSITY
Department of Mathematics

Math 102 | **Jan. 22, 2011**
Calculus II | **Final Exam** | **Time: 2 hrs.**

Use of calculators is not allowed in this exam. Please switch off your mobile phones.

1. (2+1 pts.) Let y be a function of x defined implicitly by

$$y^3 + x^2 e^y + x^4 = 1 \qquad x \geq 0$$

   (a) Show that this function is one-to-one.

   (b) Show that the graph of the inverse function has a vertical tangent at (0,1)

2. (3 pts.) Prove that $\tanh(\log_3 x) = \dfrac{x^{2/\ln 3} - 1}{x^{2/\ln 3} + 1}$.

3. (3 pts.) Evaluate the following limit

$$\lim_{x \to 3} \frac{(2^x - 3)\tan^{-1}(x-3)}{\log_3(x) - 1}$$

4. (5+4 pts.) Evaluate the following integrals

   (a) $\displaystyle\int x[\ln(1+x^2)]^2\,dx$ (b) $\displaystyle\int \frac{x^2\,dx}{\sqrt{1-x^2}+\sqrt{1+x^2}}$

5. (4 pts.) Determine whether the following improper integral is convergent or divergent. If it converges, find its value.

$$\int_{-\infty}^{-2} \frac{x+1}{x^2+2x}\,dx$$

6. (4 pts.) Find the coordinates of the centroid of the region bounded by the curves $x = 5 - y^2$ and $x = 0$.

7. (4+4+2 pts.) Let C be the curve given by the parametric equations

$$x = t - \sin t \qquad y = 1 - \cos t \qquad ;\ 0 \leq t \leq \frac{\pi}{2}$$

   (a) Find the length of C.

   (b) Find the area of the surface obtained by rotating C about the x-axis.

   (c) Find $\dfrac{d^2y}{dx^2}$

8. (2+2 pts.) Consider the curve given by the polar equation

$$r = (\pi - \theta)^2 \qquad ;\ 0 \leq \theta \leq 2\pi$$

   (a) Sketch the curve.

   (b) Find the area inside the curve.

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1. (a) $3y^2y' + x^2e^yy' + 2xe^y + 4x^3 = 0 \Longrightarrow y' = \dfrac{-(2xe^y + 4x^3)}{3y^2 + x^2e^y} < 0, \forall x > 0 \Longrightarrow f$ is decreasing $\forall x > 0 \Longrightarrow f$ is 1-1.

   (b) $\lim\limits_{x\to1^-} \left|\dfrac{d}{dx}f^{-1}(1)\right| = \lim\limits_{x\to1^-} \left|\dfrac{1}{f'(f^{-1}(1))}\right| = \lim\limits_{x\to1^-} \left|\dfrac{1}{f'(0)}\right| = \infty$.

2. $\tanh\left(\dfrac{\ln x}{\ln 3}\right) = \dfrac{\exp\left(\dfrac{\ln x}{\ln 3}\right) - \exp\left(\dfrac{-\ln x}{\ln 3}\right)}{\exp\left(\dfrac{\ln x}{\ln 3}\right) + \exp\left(\dfrac{-\ln x}{\ln 3}\right)} = \dfrac{x^{1/\ln 3} - x^{-1/\ln 3}}{x^{1/\ln 3} + x^{-1/\ln 3}} = \dfrac{x^{2/\ln 3} - 1}{x^{2/\ln 3} + 1}$

3. Indeterminate form $\frac{0}{0}$. Applying L'Hopitals rule $\Longrightarrow \lim\limits_{x\to3} \dfrac{2^x \ln 2 \tan^{-1}(x-3) + \left[\dfrac{(2^x - 3)}{(x-3)^2 + 1}\right]}{(x\ln 3)^{-1}} = 15\ln 3$

4. (a) Let $z = 1 + x^2, dz = 2xdx$, (or let $z = \ln(1+x^2) \Longrightarrow e^z = 1 + x^2, e^zdz = 2xdx$), then substituting we have $\Longrightarrow \frac{1}{2}\int(\ln z)^2 dz$, using integration by parts twice: let $u = (\ln z)^2$& $dv = dz$, then $\frac{1}{2}\int(\ln z)^2 dz = \frac{1}{2}\left[z(\ln z)^2 - 2\int \ln z dz\right]$. Then, since $\int \ln z dz = z\ln z - z + c$, we have $\frac{1}{2}\int(\ln z)^2 dz = \frac{1}{2}\left[z(\ln z)^2 - 2(z\ln z - z)\right] + c$. Hence,

$$\int x\left[\ln(1+x^2)\right]^2 dx = \frac{1}{2}(1+x^2)\left[\ln(1+x^2)\right]^2 - (1+x^2)\left[\ln(1+x^2)\right] + (1+x^2) + c$$

   (b) $\int ...dx = \int \dfrac{x^2\left(\sqrt{1-x^2} - \sqrt{1+x^2}\right)}{-2x^2}dx = \frac{-1}{2}\int\sqrt{1-x^2}dx + \frac{1}{2}\int\sqrt{1+x^2}dx = I_1 + I_2$. Solving I_1 : Let $x = \cos\theta \Longrightarrow \frac{1}{2}\int \sin^2\theta d\theta = \frac{1}{4}\int(1-\cos 2\theta)d\theta = \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right) + c_1$. Therefore, $I_1 = \frac{1}{2}\left(\cos^{-1}\theta - x\sqrt{1-x^2}\right) + c_1$. Solving I_2 : Let $x = \tan\theta \Longrightarrow I_2 = \frac{1}{2}\int\sec^3\theta d\theta = \frac{1}{2}(\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|) + c_2$. Therefore, $I_2 = \frac{1}{2}\left(x\sqrt{1+x^2} - \ln\left|\sqrt{1+x^2} + x\right|\right) + c_2$.

5. $\int\limits_{-\infty}^{-2} ...dx = \lim\limits_{t\to-\infty}\int\limits_{t}^{-1} ...dx + \lim\limits_{z\to-2}\int\limits_{-1}^{z} ...dx$. Solving the integral: let $u = x^2 + 2x, du = (2x+2)dx \Longrightarrow \frac{1}{2}\int\dfrac{du}{u} = \frac{1}{2}\ln|x^2 + 2x| + c$. Then the improper integral is divergent.

$$\int\limits_{-\infty}^{-2} ...dx = \frac{1}{2}\left[\left(\ln 3 - \lim_{t\to-\infty}\ln|t^2 + 2t|\right) + \left(\lim_{z\to-2}\ln|z^2 + 2z| - \ln 3\right)\right] = -\infty$$

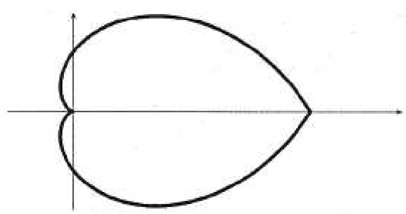
6. Let $y = \pm\sqrt{5-x}$, is symmetric about $x - axis$. $Area = 2\int\limits_0^5 \sqrt{5-x}dx = 2\left[\frac{-2}{3}(5-x)^{3/2}\right]_0^5 = \dfrac{20}{3}\sqrt{5}$. Then, $\bar{x} = \dfrac{2}{A}\int\limits_0^5 x\sqrt{5-x}dx$, let $u^2 = 5 - x \Longrightarrow \bar{x} = \dfrac{4}{A}\int\limits_0^{\sqrt{5}} u^2(5-u^2)du = \frac{4}{A}\left[\dfrac{5u^3}{3} - \dfrac{u^5}{5}\right]_0^{\sqrt{5}} = 2$. And, $\bar{y} = 0$, by symmetricity. Therefore, the centroid $= \left(\bar{x}, \bar{y}\right) = (2, 0)$.

7. (a) $L = \int_0^{\pi/2} \sqrt{(1-\cos t)^2 + \sin^2 t}dt = \sqrt{2} \int_0^{\pi/2} \frac{\sqrt{1-\cos^2 t}}{\sqrt{1+\cos t}}dt = \sqrt{2} \int_0^{\pi/2} \frac{\sin t}{\sqrt{1+\cos t}}dt \Longrightarrow$ let $u = 1 + \cos t, du = -\sin t dt$, then: $\sqrt{2} \int_0^{\pi/2} \frac{\sin t}{\sqrt{1+\cos t}}dt = -2\sqrt{2}\left[\sqrt{1+\cos t}\right]_0^{\pi/2} = -2\sqrt{2}\left(1-\sqrt{2}\right).$

(b) $S = 2\pi \int_0^{\pi/2} y ds = 2\pi \int_0^{\pi/2} (1-\cos t)\frac{\sin t}{\sqrt{1+\cos t}}dt = -4\pi\sqrt{2}\left(1-\sqrt{2}\right) + 2\pi \int_0^{\pi/2} \frac{\cos t \sin t}{\sqrt{1+\cos t}}dt$. Solving the remaining integral, let $u = 1+\cos t, du = -\sin t dt$, then $\Longrightarrow 2\pi \int_0^{\pi/2} \frac{\cos t \sin t}{\sqrt{1+\cos t}}dt = 2\pi \int_2^1 \left(u^{1/2} - u^{-1/2}\right) du = 4\pi\left[\frac{u^{3/2}}{3} - u^{1/2}\right]_2^1 = 4\pi\left(\frac{\sqrt{2}-2}{3}\right)$. Therefore, $S = \frac{4\pi}{3}\left(4-2\sqrt{2}\right).$

(c) $\frac{dy}{dx} = \frac{\sin t}{1-\cos t}$. Then, $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right)}{\frac{dx}{dt}} = \frac{-1}{(1-\cos t)^2}.$

8. (a) $r = (\pi - \theta)^2$



(b) $A = 2\left[\frac{1}{2}\int_0^{\pi} (\pi-\theta)^4 d\theta\right] = .\int_0^{\pi}\left(\theta^4 - 4\pi\theta^3 + 6\pi^2\theta^2 - 4\pi^3\theta + \pi^4\right) d\theta$

$$= \left[\frac{\theta^5}{5} - \pi\theta^4 + 2\pi^2\theta^3 - 2\pi^3\theta^2 + \pi^4\theta\right]_0^{\pi} = \frac{\pi^5}{5}$$